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SEPARATION OF A FLOW FROM THE CORNER POINT OF A BODY*

E.V. BOGDANOVA and O.S. RYZHOV

Changes in the velocity field which occur when the external pressure gradient is gradually increased, the gradient being determined by the theory of jet flows of an ideal incompressible fluid, are studied. The possibility of an essentially non-linear viscous sublayer occuring in the preseparation region, which adheres to the rigid surface, is noted. A solution of the boundary value problem is given for a boundary layer interacting freely with the potential flow under the conditions when the initial pressure gradient changes its sign from negative to positive. In this case a stagnation point appears in the incoming flow.

1. The preseparation region.

We shall assume that the surface of the streamlined body has a corner, at which the flow becomes separated. We choose the radius of curvature of the surface, the velocity of potential flow of fluid at the corner point, and its density, as the three basic units of measurement. Assuming that a change to dimensionless variables has been made, we shall direct the s axis of the curvilinear orthogonal system of coordinates along the generatrix of the body, and the n axis along its normal. Let u' and v' be the components of the perturbed velocity vectors and p' the excess pressure in the outer potential

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region of the flow. The theory of jet flows of an ideal incompressible fluid shows that near the point at which the free streamline leaves the body the complex velocity has a singularity. In the linear approximation the Bernoulli integral implies that $u' = -p'$, and from this we have /1/

$$w' = p' + iv' = ib_{1/2}z^{1/2} + \ldots, \quad z = s + in \tag{1.1}$$

The constant $b_{1/2}$ is determined by the global structure of the velocity field, and its sign can be negative, as well as positive. When $\arg z \to 0$, the excess pressure $p' \to 0$. If on the other hand $\arg z \to \pi$, then

$$p' \to -b_{1/2}(-s)^{1/2} + \ldots \tag{1.2}$$

When $b_{1/2}$ of (1.1) changes its sign, so does the derivative dp'/ds characterizing the drop or rise in pressure along the surface $s < 0$. As regards the extent of the region of preseparation flow, it is determined by the normalization $s = |b_{1/2}|\lambda^{-3}x_1$ where λ is the initial value of the reduced surface friction /2/. Further we shall assume that the relation $\Delta = |b_{1/2}|\lambda^{-1} \ll 1$ expressed in terms of the Reynolds number $R \gg 1$ results from at least one of the following inequalities: $|b_{1/2}| \ll 1$ or $1 \ll \lambda$.

The Prandtl boundary layer with the singularity (1.2) in the pressure distribution on its outer edge and $\lambda \sim 1$, was studied in /3/ in connection with the problem of the flow past a flat plate positioned at a positive angle of attack $\alpha^* = b_{1/2} \sim R^{-1/16}$ to the incoming flow. The analysis for the region of preseparation flow at the surface of a smooth body was carried out in /4/ and was subsequently adopted in /5, 6/. The structure of the classical boundary layer in another limiting case when $b_{1/2} < 0$, and $|b_{1/2}| \sim 1$, was discussed in /7/. Such a structure is characteristic for the separation of the flow from the corner point of the body surface.

The approach proposed in /2/ gives a unified treatment to various modes of flow depending on the sign of $b_{1/2}$ and the similarity parameter $K = R^{-1/2}|b_{1/2}|^{-8}\lambda^{9}$ on different bodies.

In the linear solution constructed in /2-6/ which describes the preseparation region, the correction terms are of the order of unity when $(-x_1) \sim \Delta^5$. This characteristic size is associated with the zone in which essentially non-linear effects may appear in the motion of the fluid, provided that $|b_{1/2}|$ does not become very small and the order of K does not exceed unity. We stretch, in the viscous boundary sublayer adjacent to the body, both coordinates

$$s = |b_{1/2}|^6\lambda^{-8}\bar{x}, \quad n = R^{-1/2}|b_{1/2}|^2\lambda^{-3}\bar{y} \tag{1.3}$$

and normalize, in an appropriate manner, the stream function and the pressure

$$\psi = R^{-1/2}|b_{1/2}|^4\lambda^{-5}\bar{\psi}, \quad p' = |b_{1/2}|^4\lambda^{-4}\bar{p} \tag{1.4}$$

The Prandtl equation in the new variables states that

$$\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\,\partial\bar{y}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2} = -\frac{d\bar{p}}{d\bar{x}} + \frac{\partial^3\bar{\psi}}{\partial\bar{y}^3} \tag{1.5}$$

$$\bar{\psi} = \partial\bar{\psi}/\partial\bar{y} = 0 \quad \text{when } \bar{y} = 0$$

In order to formulate the initial conditions for it, we introduce the selfsimilar variable $\xi = (-\bar{x})^{-1/3}\bar{y}$ and determine the function $f^{(1/2)}$ as the solution of the ordinary differential equation

$$\frac{d^3f^{(1/2)}}{d\xi^3} - \frac{1}{3}\xi^2\frac{d^2f^{(1/2)}}{d\xi^2} + \frac{1}{2}\xi\frac{df^{(1/2)}}{d\xi} - \frac{1}{2}f^{(1/2)} = \frac{1}{2}\operatorname{sign} b_{1/2} \tag{1.6}$$

satisfying the boundary conditions

$$f^{(1/2)} = df^{(1/2)}/d\xi = 0 \text{ when } \xi = 0 \tag{1.7}$$

and the demand that there are no terms that increase exponentially as $\xi \to \infty$. From the results of /2-6/ we obtain

$$\bar{\psi} \to \frac{1}{2}\bar{y}^2 + (-\bar{x})^{1/2}f^{(1/2)}(\xi) + \ldots, \quad \bar{p} \to -\operatorname{sign} b_{1/2}(-\bar{x})^{1/2} + \ldots \tag{1.8}$$

as $\bar{x} \to -\infty$, $\xi = \text{const}$.

The conditions at the outer edge of the viscous boundary sublayer adjacent to the wall are obtained by matching the stream function $\bar{\psi}$ and excess pressure $\bar{p}$ normalized with the help of (1.4), with the corresponding quantities for the main bulk of the boundary layer. Using the asymptotic forms $f^{(1/2)}$ for large values of the argument, we have

$$\bar{\psi} \to \frac{1}{2}\bar{y}^2 - \frac{2\Gamma(1/6)}{3\Gamma(2/3)}\operatorname{sign} b_{1/2}\bar{y}^{1/2} + A(\bar{x})\bar{y} + \ldots \quad \text{as } \bar{y} \to \infty \tag{1.9}$$

The choice of the displacement thickness A remains arbitrary, except that /2-6/

$$A \to B_{1/2}\,\mathrm{sign}\,b_{1/2}\,(-\bar{x})^{1/4} + \ldots \quad \text{as} \quad \bar{x} \to -\infty \tag{1.10}$$

and the numerical value of the constant $B_{1/2} = 2^{1/3}3^{1/3}\Gamma\,(^1/_3)$ is established by solving problem (1.6), (1.7).

As regards the pressure gradient in Eq.(1.5), in order to determine it we must match the solution for the main bulk of the boundary layer with the solution controlling the potential flow of the fluid where the transverse $n = |\,b_{1/2}\,|^6\,\lambda^{-8}y_e$ and the longitudinal coordinate have the same scale. When analysing the potential flow, it is convenient to use, instead of the stream function, the components $u' = |\,b_{1/2}\,|^4\lambda^{-4}u_e$ and $v' = |\,b_{1/2}\,|^4\lambda^{-4}v_e$ of the perturbed velocity directly. We see from the matching /2-6/ that the usual condition of non-penetration must hold

$$v_e = -K dA/d\bar{x} \quad \text{when} \quad y_e = 0,\ \bar{x} < 0 \tag{1.11}$$

which contains the similarly parameter K mentioned above.

One might get the impression, that inclusion in it of the surface friction, which depends on the Reynolds number and possesses the property that $\lambda \to \infty$ as $R \to \infty$, is artificial. We find, however, that in the boundary layer which becomes detached near the leading edge from the sharp plate protruding from the blunt body, the surface friction is governed, according to /8, 9/, by the estimate $\lambda \sim R^{1/16}b_{1/2}^{3/8}\ (b_{1/2} > 0)$. The friction increases without limit together with the Reynolds number if the quantities $b_{1/2}$ are of higher order than $R^{-1/16}$.

2. The non-linear viscous sublayer. When $b_{1/2} < 0$ and $K \ll 1$, we obtain by virtue of (1.11) the problem of flow past the initial body for the complex velocity, since $v_e\,(\bar{x},\,0) = 0$. The excess pressure is then established from its limit value at infinity. Thus we have

$$\bar{p} = -\mathrm{sign}\,b_{1/2}\,(-\bar{x})^{1/2} + \ldots = (-\bar{x})^{1/2} + \ldots \tag{2.1}$$

while the displacement thickness $A\,(\bar{x})$ in the boundary condition (1.9) remains unknown, and is found when integrating Eq.(1.5). The integration starts, as $\bar{x} \to -\infty$, from the initial distribution for the stream function given by the first equation of (1.8).

A solution of the problem formulated should be constructed using numerical methods. However, from the theory of the Prandtl boundary layer we know /10/ that the influence of initial data is appreciable at distances several times greater than the displacement thickness. Therefore, to a first approximation the initial data need not be taken into account at all when the question concerns the asymptotic properties of the solution in the limit as $\bar{x} \to 0-$. This simple concept forms the basis of /7/, and following this paper we shall separate the region of flow in question, with the longitudinal and transverse coordinates normalized according to (1.3), into two sublayers, and this will lead to a four-layer structure of the velocity field.

In order to satisfy the condition that the fluid adheres to the surface of the body, we must include in the balance of forces the influence of viscous tangential stresses, and this results in the selfsimilar form

$$\bar{\psi} = 2^{1/2}(-\bar{x})^{5/6}f(\eta), \quad \eta = 2^{-1/2}(-\bar{x})^{-1/6}\bar{y} \tag{2.2}$$

of the asymptotic solution in the lowest of the four sublayers. Substituting (2.2) into (1.5), we obtain a non-linear differential equation for the function f, represented in the limit, as $\eta \to \infty$, by the following sequence:

$$f = \sum_{m=0}^{\infty} a_m \eta^{(5-m)/3} \tag{2.3}$$
$$a_1 = a_2 = a_5 = 0,\ a_4 = 9\,(5a_0)^{-1},\ a_6 = a_3^2\,(5a_0)^{-1}$$

where the remaining coefficients a_0 and a_3 are arbitrary. Numerical integration of this equation starts at the point $\eta = 0$ where $f = df/d\eta = 0$. Comparing the data thus obtained with (2.3) we find /7/ that $a_0 = 1.950718$, and $a_3 = -1.577568$.

Let us write, using (2.3) as the starting point, the asymptotic expansion for the stream function for fixed $(-\bar{x}) \ll 1$ and $\eta \to \infty$, namely

$$\bar{\psi} = 2^{-1/3}a_0\bar{y}^{5/3} + 2^{1/6}a_3(-\bar{x})^{1/3}\bar{y}^{2/3} + 2^{1/3}a_4(-\bar{x})^{1/2}\bar{y}^{1/3} + \ldots \tag{2.4}$$

The form of expression (2.4) implies the existence of a sublayer whose properties are very similar to those of the basic bulk of the boundary layer /2-6/. Indeed, here we have

$$\bar{\psi} = \bar{\psi}_0(\bar{y}) + (-\bar{x})^{1/3}\bar{\psi}_1(\bar{y}) + (-\bar{x})^{1/2}\bar{\psi}_2(\bar{y}) + \ldots \tag{2.5}$$

The functions $\bar{\psi}_1$ and $\bar{\psi}_2$ satisfy the following linear differential equations:

$$\frac{d\bar{\psi}_0}{d\bar{y}}\frac{d\bar{\psi}_1}{d\bar{y}} - \frac{d^2\bar{\psi}_0}{d\bar{y}^2}\bar{\psi}_1 = 0, \quad \frac{d\bar{\psi}_0}{d\bar{y}}\frac{d\bar{\psi}_2}{d\bar{y}} - \frac{d^2\bar{\psi}_0}{d\bar{y}^2}\bar{\psi}_2 = -1 \tag{2.6}$$

with the coefficients expressed in terms of the arbitrary function $\bar\psi_0$. Recalling (1.9), we have the condition in the limit

$$\bar\psi_0 \to -\frac{1}{2}\bar y^2 + \frac{2\Gamma(1/6)}{3\Gamma(2/3)}\bar y^{5/2} + \ldots \quad \text{as} \quad \bar y \to \infty \tag{2.7}$$

The matching of (2.4) and (2.5) at the inner boundary of the sublayer in question leads to the requirement that

$$\bar\psi_0 \to 2^{-1/3}a_0\bar y^{5/3} + \ldots \quad \text{as} \quad \bar y \to 0 \tag{2.8}$$

Integrating the first equation of (2.6) we obtain $\bar\psi_1 = C_1 d\bar\psi_0/d\bar y$. We see from (2.7) that the complete matching of this solution with the solution specifying the structure of the basic bulk of the boundary layer in (1.9), is prevented by the lack of a term proportional to $\bar y^{1/2}$. But we can establish the asymptotic expression for the function

$$A = C_1(-\bar x)^{3/5} + \ldots, \quad C_1 = 2^{1/3}3a_3(5a_0)^{-1} \tag{2.9}$$

even when we confine ourselves to terms linear in $\bar y$ (the constant is determined by matching with (2.4)).

It remains to integrate the second equation of (2.6), whose right-hand side is generated by taking into account the excess pressure (2.1). By virtue of (2.8) its general solution

$$\bar\psi_2 = C_2\frac{d\bar\psi_0}{dy} - \frac{d\bar\psi_0}{d\bar y}\left\{\int_0^{\bar y}\left[\left(\frac{d\bar\psi_0}{dy}\right)^{-2} - 2^{2/3}\left(\frac{3}{5}\right)^2\frac{1}{(a_0y^{2/3})^2}\right]dy - 2^{2/3}3\left(\frac{3}{5}\right)^2\frac{1}{a_0^2\bar y^{1/3}}\right\}$$

has the following limit as $\bar y \to 0$:

$$\bar\psi_2 \to 2^{-1/3}3^{-1}5a_0C_2\bar y^{2/3} + 2^{1/3}3^2(5a_0)^{-1}\bar y^{1/3} + \ldots \tag{2.10}$$

Matching (2.10) to (2.4) we obtain the relations $C_2 = 0$ and $a_4 = 9(5a_0)^{-1}$, the latter being obtained, of course, in agreement with (2.3). From this we have, at the outer boundary $\bar y \to \infty$ of the sublayer in question,

$$\bar\psi_2 \to D\bar y + \ldots, \quad D = -\int_0^{\infty}\left[\left(\frac{d\bar\psi_0}{dy}\right)^{-2} - 2^{2/3}\left(\frac{3}{5}\right)^2\frac{1}{(a_0y^{2/3})^2}\right]dy$$

and this enables us to sharpen the asymptotic expression for the function A. Finally we have

$$A = C_1(-\bar x)^{3/5} + D^{\cdot}(-\bar x)^{1/2} + \ldots$$

Thus we see that the essentially non-linear nature of the motion of the fluid in the region with normalized (1.3) longitudinal and transverse coordinate, leads to a more rapid reduction compared with (1.10) in the displacement thickness A near the point where the free streamline leaves the corner point of the body. As a result, the derivative $dA/d\bar x$, which determines the angle of inclination of the velocity vector, becomes smaller.

The reduced surface friction corresponding to the solution (2.2) is $\lambda_w = 2^{-1/2}\lambda(-\bar x)^{-1/5}d^2f(0)/d\eta^2$, and this yields the estimate $s \sim |b_{1/2}|^6\lambda_w^{-8}$ which repeats, when $K \sim 1$ and $\lambda_w \sim \lambda$, the first normalization of (1.3) based on completely different concepts. However, when $K \ll 1$, the local value λ_w of the surface friction increases in the non-linear region, gradually becoming many times greater than its initial value λ. The preliminary increase in the surface friction ensures the subsequent appearance of free interaction between the boundary layer and outer potential flow.

Let us estimate the reduced coordinate $\bar x$ at which the above process starts. In the potential flow the excess pressure $p_e = |b_{1/2}|^{-4}\lambda^4p'$ is of the same order as both components u_e and v_e of the perturbed velocity. Since the pressure does not vary across all three sublayers into which the boundary layer is decomposed, it follows that $\bar p \sim p_e \sim v_e$. We shall calculate the last of these quantities using relation (1.11) and substituting it into the right-hand side of A from formula (2.9). The expression $\bar p \sim K(-\bar x)^{-3/5}$ obtained in this manner should be of the same order as the excess pressure which, by virtue of (2.1), is simply $(-\bar x)^{1/2}$. This yields $\bar x \sim R^{-4/9}|b_{1/2}|^{-64/9}\lambda^8$. The final estimate

$$-s \sim R^{-4/9}|b_{1/2}|^{-10/9} \tag{2.11}$$

does not contain $\lambda \ll R^{1/18}|b_{1/2}|^{8/9}$, and this is in full agreement with the assertion that the asymptotic properties of the flow as $\bar x \to 0$ are independent of its state at the input $\bar x \to -\infty$. In order to describe the process of free interaction itself we can use, in this case, the well-known solution given in /11, 12/.

3. Free interaction mode. Now let $K \sim 1$, from which it follows that $|b_{1/2}| \sim R^{-1/16}\lambda^{9/8}$. Under this condition the free interaction of the boundary layer with the outer potential flow

begins at once, without preliminary formation of a non-linear viscous sublayer with the known excess-pressure distribution (2.1). The extent of the region of free interaction is governed, as can be confirmed, by the estimate (2.11), although in the mode in question the initial surface friction $\lambda \sim R^{1/16}|b_{1/2}|^{8/9}$. With regard to the sign of $b_{1/2}$, we will allow the possibility that it will change from negative to positive when it is gradually increased. The passage of $b_{1/2}$ through zero means that the boundary layer in front of the break in the contour of the body begins to be affected by the adverse pressure gradient causing a drop in the surface friction. However, when considering $b_{1/2} > 0$, we shall limit ourselves to the case when the separation occurs at the corner point. The resulting non-trivial boundary value problem will be analysed in detail below.

Let us superimpose the s axis of the initial curvilinear system of coordinates behind the point of separation, with the streamline detached from this point. We introduce the complex variable $z_e = \bar{x} + iy_e$. Relation (1.1) shows that in the potential flow the excess pressure $p_e \to 0$ and $|z_e| \to \infty$ and $\arg z_e = 0$. Let the detached boundary layer separate the moving fluid from the fluid at rest. Repeating the arguments of /11/ we can show that in the this case $\bar{p} = 0$, within the approximation used, in the whole region $\bar{x} > 0$ of the boundary layer. As usual, we shall determine the analytic function w_e of the complex variable z_e, using $w_e = p_e + iv_e$. Recalling (1.11), we formulate the Hilbert problem for this function in the upper half-plane with the following data:

$$\operatorname{Im} w_e = -K dA/d\bar{x} \quad \text{when} \quad \bar{x} < 0; \ \operatorname{Re} w_e = 0 \quad \text{when} \quad \bar{x} > 0 \tag{3.1}$$

on the axis $y_e = 0$. In addition, the function w_e must satisfy, by virtue of (1.8), the limit relation $w_e \to i \operatorname{sign} b_{1/2} z_e^{1/2}$ when $|z_e| \to \infty$, irrespective of the fact that according to (1.10) the limiting value $\operatorname{Im} w_e \to {}^1/_6 \operatorname{sign} b_{1/2} K B_{1/2} (-\bar{x})^{-5/6}$ as $\bar{x} \to -\infty$.

For functions with such an order of increase at infinity the problem should have zero index, ensuring the uniqueness of the solution, provided that we had $|w_e| \sim |z_e|^m$ with ${}^1/_2 < m \leqslant {}^3/_2$ as $|z_e| \to 0$. This behaviour of w_e is quite admissible, since it implies the smoothing of the singularity which is generated in the initial flow of ideal fluid due to neglect of the tangential viscous stresses. It is, however, impossible to satisfy automatically the conditions at zero and infinity, since the asymptotic form $w_e \to i \operatorname{sign} b_{1/2} z_e^{1/2}$ as $|z_e| \to \infty$ does not correspond to the asymptotic forms of $\operatorname{Re} w_e$ and $\operatorname{Im} w_e$ appearing in the boundary conditions (3.1).

We shall begin with estimates fixing just the order of increase of w_e at both singularities. Standard methods of functions of a complex variable yield in this case

$$w_e = \frac{K}{\pi i} z_e^{3/2} \int_{-\infty}^{0} \frac{dA/d\xi}{(-\xi)^{3/2}(\xi - z_e)} d\xi \tag{3.2}$$

However, the coefficient of $z^{1/2}$ in the asymptotic expansion of w_e following from (3.2), generally speaking, different at infinity, from $i \operatorname{sign} b_{1/2}$. In order to make it equal to a given quantity, we must impose the requirement that

$$\int_{-\infty}^{0} \frac{dA/d\xi}{(-\xi)^{3/2}} d\xi = \operatorname{sign} b^{1/2} \frac{\pi}{K} \tag{3.3}$$

on the required displacement thickness A. As a result, the excess pressure in the region $\bar{x} < 0$ of the boundary layer will be expressed as

$$\bar{p} = -(-\bar{x})^{1/2} \left[\operatorname{sign} b_{1/2} + \frac{K}{\pi} \int_{-\infty}^{0} \frac{dA/d\xi}{(-\xi)^{1/2}(\xi - \bar{x})} d\xi \right] \tag{3.4}$$

Now we have formulated completely the boundary value problem for the Prandtl Eq.(1.5), but condition (3.3) is additional. It will have to be confirmed that it is satisfied. If on the other hand (3.3) is not satisfied, then the use of (3.4) to represent the excess pressure would imply that the singularity in the region of free interaction is no longer smoothed.

Instead of starting with (3.2), we can start directly from (3.4) as was done in /12/ for flows with a favourable pressure gradient satisfying $|b_{1/2}| \sim 1$. Then the special behaviour of the unique solution $|w_e|$ when $|z_e| \to 0$ would be determined by the inequalities $-{}^1/_2 < m \leqslant {}^1/_2$. Relation (3.3) guarantees that it belongs to the class of functions in question, and the relation, in this case, plays the part of the condition of solvability of the Hilbert problem formulated in the corresponding manner.

Since the Prandtl equation is non-linear, it follows that the boundary value problem for the region of free interaction can only be solved using numerical methods. We can check whether the additional condition (3.3) holds, only with an accuracy guaranteed by the finite difference scheme chosen. Therefore, it becomes very important to simplify the problem so that an exhaustive analysis can be carried out. The possibilities in this connection are related to the passage of the coefficient $b_{1/2}$ through zero.

Let us assume that, since $|b_{1/2}|$ is small, the similarity parameter K reaches fairly large values compared with unity. Let us write $b_{1/2} = kb_{1/2}^{\circ}$, where k is a positive number and, in all transformations carried out above of the independent variables as well as of the functions sought, let us replace $|b_{1/2}|$ by $|b_{1/2}^{\circ}|$. Eq.(1.5) retains its form, relations (1.8)-(1.10) will have $k\,\mathrm{sign}\,b_{1/2}^{\circ}$ instead of $\mathrm{sign}\,b_{1/2}$, and a modified similarity parameter $K^{\circ} = R^{-1/2}|b_{1/2}^{\circ}|^{-8}\lambda^{9}$ will appear on the right-hand side of (1.11). Let us choose the constant k so that $K^{\circ} = 1$. This yields the normalization $|b_{1/2}^{\circ}| = R^{-1/16}\lambda^{9/8}$, with the obvious corollary $k = |b_{1/2}|R^{1/16}\lambda^{-9/8} = K^{-1/8}$, i.e. $k \sim 1$ when $K \sim 1$, but $k \ll 1$ when $1 \ll K$. It is important that the continuation $-s \sim R^{-3/8}\lambda^{-5/4}$ of the region of free interaction is obtained, unlike (2.11), in a fixed form depending on the Reynolds number and surface friction in the initial boundary layer.

This thickness of the viscous boundary layer is estimated in terms of its order of magnitude, as $n \sim R^{-5/8}\lambda^{-3/4}$.

Thus we have formulated a boundary value problem in which the small parameter k appears in relations (1.8)-(1.10) and not in the condition of interaction (1.11). The problem can be linearized, provided that we put

$$\psi = {}^1/_2\bar{y}^2 + k\bar{\psi}'(\bar{x},\bar{y}), \quad \bar{p} = k\bar{p}'(\bar{x}), \quad A = kA'(\bar{x})$$

and its final formulation for the perturbed stream function $\bar{\psi}'$ in suitably chosen variables will be identical with that given in /3/ in connection with the determination of the antisymmetric part of the pressure acting on the plate at the angle of attack α^* to the incoming flow. Since the reduced angle $\alpha = \alpha^* R^{1/16}\lambda^{-9/8}$, we have the corresponding $k\,\mathrm{sign}\,b_{1/2}^{\circ}$. Using the results given in /3/, we can write for $\bar{x} < 0$ at once

$$\frac{d\bar{p}'}{d\bar{x}} = D_0 \int_0^{\infty} \frac{\exp(\gamma^{3/4}\bar{x}\zeta)\exp I(\zeta)}{\zeta^{1/2}(1+\zeta^{1/3})}\,d\zeta \tag{3.5}$$

$$D_0 = \frac{1}{2\pi^{1/2}}\gamma^{3/8}\,\mathrm{sign}\,b_{1/2}^{\circ}, \quad \gamma = 3^{2/3}\Gamma^{-1}\left(\frac{1}{3}\right)$$

$$I = \frac{2}{3\pi}\int_0^{\infty} \frac{\sigma^{1/3}\ln(\sigma+\zeta)}{1-3^{1/2}\sigma^{1/3}+\sigma^{2/3}}\,d\sigma$$

We have, in accordance with (3.1), $d\bar{p}'/d\bar{x} = 0$ when $\bar{x} > 0$. Assuming in (3.5) that the coordinate $\bar{x} \to 0-$, we obtain the final value of the derivative $d\bar{p}'/d\bar{x}$, which corresponds to the power index $m = 1$ in the estimate which determines the behaviour of $|w_e|$ as $|z_e| \to 0$. The change in the displacement thickness along the body contour $\bar{x} < 0$ is given by the relation

$$\frac{d^2A'}{d\bar{x}^2} = -D_0\int_0^{\infty} \frac{\exp(\gamma^{3/4}\bar{x}\zeta)\zeta^{5/6}\exp I(\zeta)}{1+\zeta^{1/3}}\,d\zeta \tag{3.6}$$

and the value of $d^2A'/d\bar{x}^2$ which follows from it in the limit as $\bar{x} \to 0$ is finite, and this yields the same index $m = 1$.

Thus, according to the solution of the linearized problem, the singularity of the form $(-\bar{x})^{1/2}$ in the pressure distribution, which is generated in the initial flow of ideal fluid by neglecting the tangential viscous stresses, is smoothed out in the zone of free interaction. During the passage through the point $\bar{x} = 0$ the pressure remains continuous, and its derivative has a first-order discontinuity. As regards the integral Eq.(3.3) in which K has been replaced by $K^{\circ} = 1$, its direct confirmation based on the formulas (3.5) and (3.6) is difficult, but arguments can be presented showing that it must hold.

Indeed, let us assume the opposite, namely that condition (3.3) does not hold. Then by virtue of (3.4) the excess pressure $\bar{p} \sim (-\bar{x})^{1/2}$ and $\bar{x} \to 0-$, and this leads to the conclusion that the pressure gradient increases without limit on approaching the corner point of the body. This obviously contradicts the earlier conclusion that the pressure gradient has a finite value near this point, and hence the relation (3.3) holds.

4. The stagnation point. The asymptotic expansions which follow from (3.5) and (3.6) as $\bar{x} \to -\infty$, show that

$$\bar{p}' = -\,\mathrm{sign}\,b_{1/2}^{\circ}\left[(-\bar{x})^{1/2} - \frac{\cos\pi/8}{2^{1/2}\gamma^{3/4}}(-\bar{x})^{-1/2} + \frac{B_{1/2}}{2\cdot 3^{1/3}}(-\bar{x})^{-5/4} + \dots\right] \tag{4.1}$$

$$A' = B_{1/2}\,\mathrm{sign}\,b_{1/2}^{\circ}(-\bar{x})^{1/4} + \dots$$

The first of these formulas shows that the initial representation (1.1) of the analytic function w' as $|z| \to 0$, must contain terms $ib_{1/2+n}z^{1/2+n}$ with not only the positive integral, but also with negative values of n. They are the eigenfunctions of the boundary value problem of a flow of ideal incompressible fluid past a break in the contour of the body. The second formula of (4.1) yields, in fact, the same value of the displacement thickness A, as that prescribed using (1.10); it controls the term proportional to $(-\bar{x})^{-5/6}$ in the expression for excess pressure. The term cannot be obtained without taking into account the tangential viscous stresses.

Returning to the initial variables, we have

$$p' = -b_{1/2}\left[(-s)^{1/2} - R^{-3/8}\lambda^{-9/4}\frac{\cos\pi/8}{2^{1/2}\gamma^{3/4}}(-s)^{-1/2} + R^{-1/2}\lambda^{-5/3}\frac{B_{1/2}}{2\cdot 3^{3/2}}(-\bar{s})^{-5/6} + \ldots\right] \tag{4.2}$$

and this yields $b_{-3/2} \sim kR^{-7/16}\lambda^{-1/8}$. Since the longitudinal component of the velocity vector $u = 1 + u'$, and by virtue of the Bernoulli integral we have $u' = -p'$, it therefore follows from (4.2) that in the outer potential flow with constant $b_{1/2} > 0$ a stagnation point must be situated at the body surface at a distance

$$-s = {}^1/_2\, k^2 R^{-7/8}\lambda^{-1/4}\gamma^{-3/2}\cos^2\pi/8$$

from the break /3/. When $k \ll 1$ it vanishes under the influence of the viscosity. If on the other hand $b_{1/2} < 0$, then a stagnation point will not form at the surface of the body.

Finally, let us turn to the Brillouin-Villat condition characterizing the limiting case $b_{1/2} = 0$. The resulting boundary value problem is very similar to that studied in /13/ in connection with the separation of the flow from a step formed by the segments of two straight lines. Let us write, for simplicity, $\lambda \sim 1$. According to the solution constructed above all the redundant quantities vanish in the region of free interaction. The next term with positive n in the singular expansion (1.1) will be $ib_{3/2}z^{3/2}$ where $b_{3/2} \sim 1$, and the excess pressure at the surface of the body will be established in this case as $p' = p_{3/2}' \sim (-s)^{3/2}$. The extent of the region of free interaction is $s \sim R^{-3/8}$, therefore we have here $p_{3/2}' \sim R^{-9/16}$, which is much greater than $R^{-1/4}$, which determines the order of the selfinduced pressure. Thus, in the limiting case the non-linear structure including three fluid sublayers with different properties will not develop near the break in the contour of the body.

When $b_{1/2} = 0$, the main contribution towards the excess pressure at distances $s \sim R^{-3/8}$ is given, according to /13/, by the characteristic solution $ib_{-1/2}z^{-1/2}$, but unlike (4.2) the constant $b_{-1/2} \sim R^{-1/2}$. Denoting this contribution by $p_{-1/2}'$, we have $p_{-1/2}' < R^{-5/16}$, which is also insufficient for the appearance of a non-linear three-layer flow.

Apart from the region of free interaction we have, in the immediate neighbourhood of the break point, a zone in which the motion of a viscous fluid obeys the complete system of Navier-Stokes equations. The scales of this zone in the longitudinal and transverse directions are both estimated to be equal to $R^{-3/4}$, and the characteristic pressure induced by the displacement layer must be of the order of $R^{-1/2}$. Continuing the excess pressure generated by the terms $ib_{3/2}z^{3/2}$ and $ib_{-1/2}z^{-1/2}$ from the expansion (1.1) to distances as short as desired from the corner point yields, respectively, $p'_{3/2} \sim R^{-9/8}$ and $p_{-1/2}' \sim R^{-1/2}$. From this we conclude that the term $ib_{3/2}z^{3/2}$ generates small corrections to the solution only in the zone of the complete Navier-Stokes equations, while taking into account the term $ib_{-1/2}z^{-1/2}$ leads to the need to consider also the region of free interaction in which the equations of the boundary layer can be linearized.

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ANOMALOUS HYDRODYNAMIC FLUCTUATIONS DURING THE DEVELOPMENT OF THERMAL CONVECTION*

O.A. GRECHANNYI and V.V. TOKARCHUK

The spectral functions of the fluctuations of the hydrodynamic variables in an inhomogeneously heated liquid in the region of Rayleigh numbers close to the threshold of convective stability have been calculated using the equations of the correlation theory of thermal fluctuations in non-equilibrium statistical systems. It is shown that anomalous fluctuations, which form the structure of the flow in the regions of supercritical values of the Rayleigh number are of an essentially non-equilibrium nature and are completely accounted for by the long wavelength part of the correlation functions. The results of the calculations are used to analyse the effect of large-scale fluctuations on the Rayleigh scattering of radiation. It is is shown that, in a region where thermal convection develops, they are responsible for a phenomenon which is analogous to the critical opalescence of light during equilibrium phase transitions of the second kind.

The investigation of the dependence of the spectral functions of thermal hydrodynamic fluctuations on the degree of non-equilibrium in statistical systems is of great significance in the development of optical methods for the noise diagnostics of inhomogeneous flows of liquids and gases. Systems which are far removed from thermodynamic equilibrium and, in particular, the fluctuation mechanisms of processes involving the self-organization of flow structures when there is loss of stability are of special interest. A large number of papers (/1-4/, for example) have been concerned with the study of the anomalous hydrodynamic fluctuations which develop close to the thermal convection threshold in a liquid which is heated from below. However, the results obtained in the majority of these papers are contradictory as for example, in /1/ and /2/. This is explained by the previously discussed /5-7/ incompleteness of the theories of non-equilibrium hydrodynamic fluctuation theories which were employed. In this paper an analysis of the anomalous fluctuations during the development of thermal convection is carried out using the solution of the equations of the theory in /6/ which enables one to evaluate the results which have previously been obtained from common positions.

1. Initial equations and formulation of the problem. Let us consider a one-component inhomogeneous continuous medium which is described by a system of Navier-Stokes-Fourier equations for the mean values of the density n, the hydrodynamic velocity $\mathbf{u}$ and the thermal energy density $e = {}^3/_2\, k_B T$, where k_B is Boltzmann's constant and T is the mean value of the temperature. We shall write this system of equations in the symbolic form:

$$\frac{\partial}{\partial t}\Phi_\nu + \Lambda_\nu[\Phi; \mathbf{r}] = 0, \quad \nu = 0, 1, 2, 3, 4 \tag{1.1}$$

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